An Introduction to Superstring Theory
and its Duality Symmetries

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In these proceedings for the First School on Field Theory and Gravitation (Vitória, Brasil), a brief introduction is given to superstring theory and its duality symmetries. This introduction is intended for beginning graduate students with no prior knowledge of string theory.

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1. Introduction and Motivation

These proceedings are based on two lectures presented in April 1997 at the First School on Field Theory and Gravitation in Vitória, Brasil. Instead of giving references in these proceedings, I will refer the reader to the following review articles which appeared recently on the hep-th bulletin board: hep-th 9612254 (a review of perturbative string theory by H. Ooguri), hep-th 9702201 (an introduction to string dualities by C. Vafa), hep-th 9607021 (lectures on superstring and M-theory dualities by J. Schwarz), hep-th 9612121 (four lectures on M-theory by P. Townsend), hep-th 9609176 (a lecture on T and S-dualities by A. Sen), hep-th 9611050 (an introduction to D-branes by J. Polchinski), and hep-th 9611203 (a review of supermembranes by M. Duff). There are also two colloquia for a general audience which are available on the hep-th bulletin board: hep-th 9607067 by J. Schwarz and hep-th 9607050 by J. Polchinski.

There are various motivations for studying superstring theory, both mathematical and physical. Since I am a physicist, I will only mention the physical motivations. When string theory was discovered in the early 1970's, it was originally intended to be a model for describing strong interactions. The basic discovery was that by extending the pointlike nature of particles to one-dimensional extended objects called strings, one could obtain S-matrix scattering amplitudes for the fundamental particles which contained many of the properties found in scattering experiments of mesons. As will be discussed in section 2, the action for string theory is proportional to the area of a two-dimensional worldsheet, as opposed to the action for point-particles which is based on the length of a one-dimensional worldline.

Amazingly, the masses and coupling constants of the fundamental particles in string theory are not inputs in the theory, but are instead fixed by consistency requirements such as Lorentz invariance and unitarity. In fact, unlike theories based on point particles, string theory not only predicts the masses of the fundamental particles, but also predicts the dimension of spacetime. In the simplest string theory, this dimension turns out to be 26, rather than the experimentally observed spacetime dimension of 4. However, as will be discussed in section 5, it is possible to 'compactify' all but four of the dimensions to small circles, in which case only four-dimensional spacetime is observable at low energies.

For open string theory (where particles are represented by one-dimensional objects with two ends), the particle spectrum contains a massless 'gluon', as well as an infinite number of massive particles whose masses and spins sit on 'Reggae trajectories'. These

Reggae trajectories of massive particles are welcome for describing strong interactions since they are needed for producing scattering amplitudes with the properties seen in experiments. Unfortunately, string theory also predicts fundamental particles which are not needed for describing strong interactions. One of these particles is tachyonic, i.e. its $(\text{mass})^2$ is negative implying that it travels faster than the speed of light. The presence of such a particle makes the vacuum unstable, which is not acceptable in a physical theory.

The resolution of this tachyon problem was found in a series of remarkable discoveries which led to the concept of supersymmetry, a symmetry relating bosonic and fermionic particles. The first discovery was the existence of a new consistent string theory whose spacetime dimension turns out to be 10 rather than 26. The second discovery was that the action for this new string theory depends on a two-dimensional worldsheet containing both bosonic and fermionic parameters, and the action is invariant under a worldsheet supersymmetry which transforms the bosonic and fermionc parameters into each other. The third discovery was that, after performing a projection operation which removes half the particles but leaves a unitary S-matrix, the particle spectrum and interactions of this 'superstring' theory are invariant under a ten-dimensional spacetime-supersymmetry which transforms bosons into fermions. This projection operation removes the problematic tachyon from the spectrum but leaves the massless gluons, as well as an infinite number of massive particles. Superstring theory also contains fermionic counterparts to the gluon (called the gluino), as well as an infinite number of massive fermions.

Another particle which survives the projection operation is a massless spin-two particle called the graviton (as well as its fermionic counterpart, the massless spin-3/2 particle called the gravitino). Although this massless spin-2 particle comes from closed string theory (where particles are represented by one-dimensional circles), unitarity implies that the two ends of an open string can join to form a closed string, so these massless spin-two particles are produced in the scattering of gluons. Since the only consistent interactions of massless spin-two particles are gravitational interactions, string theory 'predicts' the existence of gravity. Therefore, without prior intention, superstring theory was found to give a unified description of Yang-Mills and gravitational interactions.

Since the energy scale of gravitational interactions is much larger than the energy scale of strong interactions, a unification of these interactions implies that the massive particles predicted by superstring theory contain masses of the order of the Planck mass (about 10^{19} GeV), and are therefore unrelated to meson particles found in experiments. So the original motivation for using string theory as a model for strong interactions is no

longer viable, assuming that one interprets the massless spin-two particle as the graviton of general relativity. Instead, superstring theory can be used as a model for a unified theory which includes all four of the standard interactions: gravitational, strong, weak, and electromagnetic (the last three are described by a spontaneously broken Yang-Mills theory).

The usual obstacle to constructing a quantum unified theory (or even a quantum theory of gravity) is that the Einstein-Hilbert action for general relativity is non-renormalizable. This is easily seen from the fact that the gravitational coupling constant (Newton's constant) is dimensionful, unlike the coupling constant of Yang-Mills theory. So for a scattering amplitude of three gravitons at L loop-order, power counting arguments imply that the amplitude diverges like Λ^{2L} where Λ is the cutoff. The only way to remove this divergence is if there is some miraculous cancellation of Feynmann diagrams.

One way to cancel divergences in Feynmann diagrams is to introduce fermions into the theory with the same interactions and masses as the bosons. Since internal loops of fermions contribute with an extra minus sign as compared with internal loops of bosons, there is a possibility of cancellations. If a theory is supersymmetrized (i.e. fermions are introduced in such a manner that the theory is symmetric under a transformation which exchanges the bosons and fermions), then the above conditions are satisfied. The supersymmetrization of gravity is called supergravity, and for a few years, it was hoped that such a theory might be free of non-renormalizable divergences. However, it was later realized that even after supersymmetrizing gravity to a theory with the maximum number of supersymmetries (which is called N=8 supergravity), the non-renormalizable divergences are still present.

As already mentioned, the fundamental particles of superstring theory include the graviton and the gravitino (like supergravity), but also include an infinite set of massive bosons and fermions. It turns out that after including the contributions of the infinite massive particles, the non-renormalizable divergences in the loop amplitudes completely cancel each other out. Although the explicit proof of the preceding statement is rather technical, there are various 'handwaving' arguments which are convincing. One of these arguments involves the nature of superstring interactions which are 'smoother' than the interactions of point-particles. For example, the three-point diagram for point-particles has a vertex where the three external point-particles coincide. But the three-point diagram for closed strings is like a pair of pants, where the two cuffs and the waist are the external

strings. Unlike the vertex in a point-particle diagram, there is no singular point on a pair of pants.

So superstring theory provides a consistent theory of quantum gravity which, unlike all other attempts, does not suffer from non-renormalizable divergences. However, it requires an infinite set of massive particles which are unobservable in any foreseeable experiment. In addition, the theory includes a set of massless particles such as the gluons and gluinos of super-Yang-Mills and also a scalar massless boson called the dilaton. If superstring theory really describes nature (and is not just a model for a unified quantum theory of gravity and Yang-Mills), these massless particles must become the leptons, quarks, and gluons of the standard model where the masses of the above particles come from spontaneous symmetry breaking. One important unsolved problem in superstring theory is that it is very difficult to give a mass to the dilaton in a natural way, so one needs to explain why noone has observed massless scalars in experiments.

Although superstring theory is the only candidate for a renormalizable quantum theory of gravity, only a few researchers worked in this field between 1975 (when it was realized that string theory could not serve as a model for strong interactions) and 1985. One reason for the lack of interest was that there appeared to be different versions of superstring theory (called Type I, Type IIA and Type IIB), none of which resembled very closely the structure of the standard model. In the Type I theory, the gauge group for super-Yang-Mills was thought to be arbitrary, and in the Type IIA and Type IIB theories, the gauge group had to be abelian. However, in 1985, it was learned that absence of anomalies restricted the gauge group of the Type I theory to be $SO(32)/Z_2$. Although this gauge group is not very interesting for phenomenology, it was soon realized that there is another type of superstring theory, called the 'heterotic' superstring (since it combines features of the bosonic string and superstring), which has two possible gauge groups: $SO(32)/Z_2$ or $E_8 \times E_8$ (E_8 is one of the exceptional groups). The $E_8 \times E_8$ version of the heterotic superstring was very attractive for phenomenologists since it is easy to construct grand unified theories starting from the exceptional subgroup E_6.

For this reason (and because of peer pressure), the next five years attracted many researchers into the field of superstring theory. However, it was soon clear that without understanding non-perturbative effects, superstring theory would not be able to give explicit predictions for a grand unified model (other than vague predictions, such as supersymmetry at a suitably high energy scale). The problem was that four-dimensional physics depends crucially on the type of compactification which is used to reduce from

ten to four dimensions. Although there is a symmetry called T-duality which relates some compactifications in superstring theory, there is a large class of compactifications which are not related by any symmetry. In principle, the type of compactification is determined dynamically, however, the selection of the correct compactification scheme requires non-perturbative information. So, for this reason (and because of problems in finding jobs), many researchers left the field of string theory after 1989 to work in other areas such as supercollider phenomenology.

Recently, it has been learned that many non-perturbative features of four-dimensional supersymmetric Yang-Mills theories can be understood without performing explicit instanton computations. Although this had been conjectured in 1977 for N=4 super-Yang-Mills, the conjecture was treated skeptically until 1994 when convincing evidence was presented for the case of N=2 super-Yang-Mills. One of these non-perturbative features is an 'S-duality' symmetry which relates the super-Yang-Mills theory at large values of the coupling constant with a super-Yang-Mills theory at small values of the coupling constant. For N=4 super-Yang-Mills, S-duality maps the theory at strong coupling into the same theory at weak coupling, while for N=2 super-Yang-Mills, S-duality maps the theory at strong coupling into a different theory at weak coupling.

These S-duality symmetries are also believed to be present in superstrings and relate superstring theory at large values of the coupling constant with a theory at small values of the coupling constant. S-duality maps the Type IIB superstring at strong coupling into the same Type IIB superstring at weak coupling, and maps the Type I superstring at strong/weak coupling into the heterotic superstring at weak/strong coupling with gauge group $SO(32)/Z_2$.

There is also believed to a duality symmetry which maps the Type IIA superstring at strong coupling into a new eleven-dimensional theory called M-theory, and which maps the heterotic superstring with gauge group $E_8 \times E_8$ at strong coupling into a version of M-theory with boundaries. M-theory is known to contain the massless particle of eleven-dimensional supergravity (which is the maximum possible dimension for supergravity) as well as massive particles which are still not understood. It is believed to be related to a theory constructed from two-dimensional extended objects called membranes (as opposed to the one-dimensional extended objects called strings).

So by studying the perturbative regime of superstring theory where the coupling constant is small, one can use S-duality symmetry to obtain non-perturbative information where the coupling constant is large. Furthermore, duality symmetries relate the five

different superstring theories, suggesting that these five theories can be understood as perturbative vacua of some unique underlying non-perturbative theory which would be the 'Theory of Everything'. This has attracted renewed interest in superstring theory, and there is optimism that by studying M-theory, one will gain a greater understanding of duality symmetries. However, the problem of getting explicit predictions out of superstring theory is probably still far from being resolved. Although S-duality symmetries may help in understanding superstring theory at very small and very large values of the coupling constants, it is not clear if it will be possible to extrapolate these results to the physically interesting values of the coupling constants which is somewhere between the two extremes.

In section 2 of this paper, I will discuss classical relativistic strings. In section 3, I will show how to quantize the relativistic string and compute the spectrum. In section 4, I shall introduce the Type IIA and Type IIB superstrings in the light-cone Green-Schwarz approach. In section 5, I shall discuss compactification and T-duality. In section 6, I will describe eleven-dimensional supergravity and give a simple argument for the S-duality symmetry of the Type IIB superstring.

2. Classical Relativistic Strings

As is well known, the action for relativistic point-particles moving in D dimensions is given by

$$S = M \int_{\tau_I}^{\tau_F} d\tau L(x^\mu(\tau)) = M \int_{\tau_I}^{\tau_F} d\tau \sqrt{\partial_\tau x^\mu \partial_\tau x_\mu} \tag{2.1}$$

and the equation of motion is $M\partial_\tau(\partial_\tau x^\mu/\sqrt{(\partial_\tau x)^2}) = 0$. In the above action, $\mu = 0$ to $D-1$, M is a dimensionful constant, and $L(x)$ is defined as the length of the path traversed by $x^\mu(\tau)$ between the times τ_I and τ_F. The momentum is defined by

$$P_\mu = M\frac{\partial L}{\partial(\partial_\tau x^\mu)} = M\frac{\partial_\tau x_\mu}{\sqrt{(\partial_\tau x)^2}}, \tag{2.2}$$

so $P_\mu P^\mu = M^2$ where M is identified with the mass of the particle.

The above action is invariant under reparameterizations of the worldline, $\tau \to \tilde{\tau}(\tau)$, allowing the gauge choice $\partial_\tau x^\mu \partial_\tau x_\mu = 1$. In this gauge, the equation of motion becomes $M\partial_\tau^2 x^\mu = 0$, which has the solution

$$x^\mu(\tau) = x_0^\mu + \tau P^\mu$$

where $P^\mu P_\mu = M^2$.

For a relativistic one-dimensional object with the topology of a closed string (i.e. the topology of a circle), the obvious generalization of (2.1)is

$$S = \frac{T}{2\pi} \int_{\tau_I}^{\tau_F} d\tau \oint_0^{2\pi} d\sigma A(x^\mu(\tau, \sigma)) \tag{2.3}$$

$$= \frac{T}{2\pi} \int_{\tau_I}^{\tau_F} d\tau \oint_0^{2\pi} d\sigma \sqrt{(\partial_\tau x^\mu \partial_\tau x_\mu)(\partial_\sigma x^\mu \partial_\sigma x_\mu) - (\partial_\tau x^\mu \partial_\sigma x_\mu)^2}$$

where T is a dimensionful constant, σ is a parameter ranging from 0 to 2π which measures the position on the circle, $\partial_\sigma x^\mu = \partial x^\mu / \partial \sigma$, and $A(x)$ is the area of the cylindrical surface traversed by $x^\mu(\tau, \sigma)$ between the times τ_I and τ_F. (The formula for the area is easily found by dividing the surface into infinitesimal parallelograms whose sides are given by $d\tau \partial_\tau x^\mu$ and $d\sigma \partial_\sigma x^\mu$.)

This action is invariant under reparameterization of the worldsurface, $\tau \to \tilde{\tau}(\tau, \sigma)$ and $\sigma \to \tilde{\sigma}(\tau, \sigma)$, which allows one to choose the gauge $\partial_\tau x^\mu \partial_\tau x_\mu = \partial_\sigma x^\mu \partial_\sigma x_\mu$ and $\partial_\tau x^\mu \partial_\sigma x_\mu = 0$. In this gauge, it is easy to show that the equation of motion from (2.3)is

$$\partial_\tau^2 x^\mu = \partial_\sigma^2 x^\mu \tag{2.4}$$

and the momentum is defined by

$$P_\mu = \frac{T}{2\pi} \frac{\partial A}{\partial(\partial_\tau x^\mu)} = \frac{T}{2\pi} \oint_0^{2\pi} d\sigma \partial_\tau x_\mu. \tag{2.5}$$

So the general solution to the equation of motion is

$$x^\mu(\tau, \sigma) = x_0^\mu + \frac{1}{T}\tau P^\mu + \sum_{N=-\infty, N\neq 0}^{\infty} (a_N^\mu e^{iN(\tau+\sigma)} + \tilde{a}_N^\mu e^{iN(\tau-\sigma)}). \tag{2.6}$$

The gauge-fixing conditions

$$\partial_\tau x^\mu \partial_\tau x_\mu - \partial_\sigma x^\mu \partial_\sigma x_\mu = \partial_\tau x^\mu \partial_\sigma x_\mu = 0 \tag{2.7}$$

imply that two of the D components of x^μ can be related to the other $D-2$ components and that

$$P^\mu P_\mu = T^2 \sum_{N=-\infty, N\neq 0}^{\infty} N^2 (a_N^j a_{-N}^j + \tilde{a}_N^j \tilde{a}_{-N}^j) \tag{2.8}$$

where $j=1$ to $D-2$.

Since $P^\mu P_\mu$ gives the (mass)^2 of the string, the mass of the string depends on the a_N^j and $\tilde{a}_N^j$ variables, and therefore depends on the way that the string is resonating. Each distinct resonance of the string corresponds to a different particle whose mass can be computed from (2.8). Although the classical relativistic string has a continuous mass spectrum, the spectrum will become discrete after quantization.

Note that T corresponds to the tension of the string since (2.4)and (2.5)imply that $\partial_\tau \hat{P}_j = T\partial_\sigma^2 x^j$ where $\hat{P}_j$ is the momentum density (i.e. $P_j = \frac{1}{2\pi} \oint_0^{2\pi} d\sigma \hat{P}_j$). In natural units for describing gravitational interactions, T is approximately $(10^{19} GeV)^2$. .

3. Quantization of the Closed String

In the previous section, it was seen that

$$\hat{P}^\mu(\tau,\sigma) = T\partial_\tau x^\mu = P^\mu + iT \sum_{N=-\infty,N\neq 0}^{\infty} N(a_N^\mu e^{iN(\tau+\sigma)} + \tilde{a}_N^\mu e^{iN(\tau-\sigma)}). \tag{3.1}$$

Using (2.6)and the canonical commutation relations

$$[x^\mu(\tau,\sigma), \hat{P}^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma - \sigma'), \tag{3.2}$$

one finds that a_N^μ and $\tilde{a}_N^\mu$ satisfy the commutation relations

$$[a_M^\mu, a_N^\nu] = \frac{1}{TN}\delta_{M+N,0}\eta^{\mu\nu}, \quad [\tilde{a}_M^\mu, \tilde{a}_N^\nu] = \frac{1}{TN}\delta_{M+N,0}\eta^{\mu\nu}. \tag{3.3}$$

As in the harmonic oscillator, one can define a ground state $|0\rangle$ which is annihilated by a_N^μ and $\tilde{a}_N^\mu$ for $N < 0$.

So using (2.8), the state

$$|\Phi\rangle = \prod_{j=1}^{D-2} \prod_{N>0} (a_N^j)^{n_N^j} (\tilde{a}_N^j)^{\tilde{n}_N^j} |0\rangle \tag{3.4}$$

has (mass)^2 given by the formula of (2.8),

$$M^2 = T^2 \langle\Phi| \sum_{N>0} N^2(2a_N^j a_{-N}^j + [a_{-N}^j, a_N^j] + 2\tilde{a}_N^j \tilde{a}_{-N}^j + [\tilde{a}_{-N}^j, \tilde{a}_N^j])|\Phi\rangle. \tag{3.5}$$

Plugging (3.4)into (3.5)and using the commutation relations of (3.3), one finds

$$M^2 = 2T \sum_{N>0} \sum_{j=1}^{D-2} N(n_N^j + \tilde{n}_N^j) + 2T \sum_{N>0} N(D-2) \tag{3.6}$$

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where the second term comes from normal-ordering (as in the ground-state energy of the harmonic oscillator).

To compute this normal-ordering term, one uses zeta-function regularization to remove the divergence. This means defining $\sum_{N>0} N$ as the analytic continuation as $s \to -1$ of $\sum_{N>0} N^{-s}$. This analytic continuation gives $\sum_{N>0} N = -\frac{1}{12}$, so the normal-ordering term (which gives M^2 for the ground-state) is equal to $2T(2-D)/12$. So when $D > 2$, the ground-state has negative (mass)^2 and is tachyonic as described in the introduction. This means that the vacuum is unstable, implying that closed string theory is inconsistent. As shown in the following section, this inconsistency is not present in closed superstring theory.

Note that the spin-two state in closed string theory is described by $|\Phi^{\mu\nu}\rangle = (a_1^\mu \tilde{a}_1^\nu + a_1^\nu \tilde{a}_1^\mu)|0\rangle$, which has $M^2 = 2T(26-D)/12$ using the formula of (3.6). So when $D = 26$ this spin-two state is massless and describes a graviton.

4. Type II Superstrings in the Light-Cone Green-Schwarz Approach

There are many equivalent descriptions of the Type IIA and Type IIB superstrings, but the only one which will be described in these notes is the formalism of Green and Schwarz in the light-cone gauge. Light-cone gauge means that the constraints of (2.7) have been used to eliminate two of the spacetime variables so one is left with the variables $x^j(\tau, \sigma)$ for $j = 1$ to $D - 2$. Unitarity and Lorentz invariance imply that $D = 10$ for the superstring, so j takes the values 1 to 8.

However, unlike the string theory of the preceding section, the Type II superstring also contains fermionic variables, $\theta^\alpha(\tau, \sigma)$ and $\tilde{\theta}^A(\tau, \sigma)$, where α and $\dot{\alpha}$ are the chiral and anti-chiral eight-dimensional spinor representations of SO(8), A is in the $\dot{\alpha}$ representation for the Type IIA superstring, and A is in the α representation for the Type IIB superstring. The chiral and anti-chiral eight-dimensional spinor representations, α and $\dot{\alpha}$, are defined using SO(8) Pauli matrices, $\sigma_{\alpha\dot{\alpha}}^j$, which satisfy the anti-commutation relations

$$\sigma_{\alpha\dot{\alpha}}^j \sigma_{\alpha\beta}^k + \sigma_{\alpha\dot{\alpha}}^k \sigma_{\alpha\beta}^j = 2\delta^{jk}\delta_{\dot{\alpha}\dot{\beta}}, \quad \sigma_{\alpha\dot{\alpha}}^j \sigma_{\beta\dot{\alpha}}^k + \sigma_{\alpha\dot{\alpha}}^k \sigma_{\beta\dot{\alpha}}^j = 2\delta^{jk}\delta_{\alpha\beta} \tag{4.1}$$

($j = 1$ to 8, $\alpha = 1$ to 8, and $\dot{\alpha} = 1$ to 8). Note that α and $\dot{\alpha}$ resemble the two-component spinor representations of SO(3,1), however in the case of SO(8), they are independently real ($(\alpha)^* = \alpha$ and $(\dot{\alpha})^* = \dot{\alpha}$) as opposed to the case of SO(3,1) where $(\alpha)^* = \dot{\alpha}$. Also, SO(8) spinor indices can be raised and lowered using the trivial metric $\delta^{\alpha\beta}$ and $\delta^{\dot{\alpha}\dot{\beta}}$.

In light-cone gauge, the action for the Type II superstring is given by

$$S = \frac{T}{2\pi} \int_{\tau_I}^{\tau_F} d\tau \oint_0^{2\pi} d\sigma (\partial_+ x^j \partial_- x^j + \theta^\alpha \partial_- \theta^\alpha + \tilde{\theta}^A \partial_+ \tilde{\theta}^A) \qquad (4.2)$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$.

The equations of motion for x^j are the same as before, and the equations of motion for θ^α and $\tilde{\theta}^A$ are $\partial_- \theta^\alpha = \partial_+ \tilde{\theta}^A = 0$, which has the general solution

$$\theta^\alpha(\tau, \sigma) = \sum_N b_N^\alpha e^{iN(\tau+\sigma)}, \quad \tilde{\theta}^A(\tau, \sigma) = \sum_N \tilde{b}_N^A e^{iN(\tau-\sigma)}. \qquad (4.3)$$

The anti-commutation relations

$$\{\theta^\alpha(\tau, \sigma), T\theta^\beta(\tau, \sigma')\} = \delta^{\alpha\beta}\delta(\sigma - \sigma'), \quad \{\tilde{\theta}^A(\tau, \sigma), T\tilde{\theta}^B(\tau, \sigma')\} = \delta^{AB}\delta(\sigma - \sigma') \qquad (4.4)$$

imply that

$$\{b_M^\alpha, b_N^\beta\} = T^{-1}\delta^{\alpha\beta}\delta_{M+N,0} \quad, \quad \{\tilde{b}_M^A, \tilde{b}_N^B\} = T^{-1}\delta^{AB}\delta_{M+N,0}. \qquad (4.5)$$

For the Type II superstring, the ground state $|0\rangle$ is defined to be annihilated by a_N^j, $\tilde{a}_N^j$, b_N^α and $\tilde{b}_N^A$ for $N < 0$. To determine the spectrum, one uses the superstring version of the gauge-fixing constraints of (2.7) which implies that the spectrum for the superstring is given by

$$M^2 = P^\mu P_\mu = T^2 \sum_{N=-\infty, N\neq 0}^{\infty} [N^2(a_N^j a_{-N}^j + \tilde{a}_N^j \tilde{a}_{-N}^j) + N(b_N^\alpha b_{-N}^\alpha + \tilde{b}_N^A \tilde{b}_{-N}^A)]. \qquad (4.6)$$

For a state

$$|\Phi\rangle = \prod_{j=1}^{8} \prod_{N>0} (a_N^j)^{n_N^j} (\tilde{a}_N^j)^{\tilde{n}_N^j} \prod_{\alpha,A=1}^{8} \prod_{M\geq 0} (b_N^\alpha)^{m_M^\alpha} (\tilde{b}_M^A)^{\tilde{m}_M^A} |0\rangle, \qquad (4.7)$$

the (mass)^2 is

$$M^2 = T^2 \langle\Phi| \sum_{N\geq 0} [N^2(2a_N^j a_{-N}^j + [a_{-N}^j, a_N^j] + 2\tilde{a}_N^j \tilde{a}_{-N}^j + [\tilde{a}_{-N}^j, \tilde{a}_N^j]) \qquad (4.8)$$

$$+ N(2b_N^\alpha b_{-N}^\alpha - \{b_{-N}^\alpha, b_N^\alpha\} + 2\tilde{b}_N^A \tilde{b}_{-N}^A - \{\tilde{b}_{-N}^A, \tilde{b}_N^A\})]|\Phi\rangle.$$

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Since

$$\sum_j N^2[a^j_{-N}, a^j_N] + \sum_j N^2[\tilde{a}^j_{-N}, \tilde{a}^j_N] = \sum_\alpha N\{b^\alpha_{-N}, b^\alpha_N\} + \sum_A N\{\tilde{b}^A_{-N}, \tilde{b}^A_N\},$$

the normal-ordering contribution from the a^j_N and $\tilde{a}^j_N$ modes is precisely cancelled by the normal-ordering contribution from the b^α_N and $\tilde{b}^A_N$ modes. Therefore, the superstring ground-state has zero mass and the excited states carry

$$M^2 = 2T \sum_{N>0} \sum_{j,\alpha,A=1}^8 N(n^j_N + \tilde{n}^j_N + m^\alpha_N + \tilde{m}^A_N). \tag{4.9}$$

Actually, there is more than one massless state of the superstring since hitting $|0\rangle$ with b^α_0 and/or $\tilde{b}^A_0$ does not change the mass. Since b^α_0 and $\tilde{b}^A_0$ satisfy the same anti-commutation relations as the SO(8) Pauli matrices in (4.1), the massless state is not a scalar of SO(8) but is actually a 256-component multiplet of SO(8). This multiplet is described by the states $|0\rangle^{jk}$, $|0\rangle^{j\dot{A}}$, $|0\rangle^{\dot{\alpha}k}$, and $|0\rangle^{\dot{\alpha}\dot{A}}$ where j, k are SO(8) vector representations and $\dot{A}$ is the opposite spinor representation of A.

The action of b^α_0 and $\tilde{b}^A_0$ on these states is defined by

$$b^\alpha_0 |0\rangle^{jk} = \sigma^{\alpha\dot{\alpha}}_j |0\rangle^{\dot{\alpha}k}, \quad b^\alpha_0 |0\rangle^{j\dot{A}} = \sigma^{\alpha\dot{\alpha}}_j |0\rangle^{\dot{\alpha}\dot{A}}, \tag{4.10}$$

$$b^\alpha_0 |0\rangle^{\dot{\alpha}k} = \sigma^{\alpha\dot{\alpha}}_j |0\rangle^{jk}, \quad b^\alpha_0 |0\rangle^{\dot{\alpha}\dot{A}} = \sigma^{\alpha\dot{\alpha}}_j |0\rangle^{j\dot{A}},$$

$$\tilde{b}^A_0 |0\rangle^{jk} = \sigma^{A\dot{A}}_j |0\rangle^{j\dot{A}}, \quad \tilde{b}^A_0 |0\rangle^{j\dot{A}} = \sigma^{A\dot{A}}_k |0\rangle^{jk}$$

$$\tilde{b}^A_0 |0\rangle^{\dot{\alpha}k} = \sigma^{A\dot{A}}_k |0\rangle^{\dot{\alpha}\dot{A}}, \quad \tilde{b}^A_0 |0\rangle^{\dot{\alpha}\dot{A}} = \sigma^{A\dot{A}}_k |0\rangle^{\dot{\alpha}k}.$$

Note that b^α_0 and $\tilde{b}^A_0$ are anti-commuting, so $|0\rangle^{jk}$ and $|0\rangle^{\dot{\alpha}\dot{A}}$ are bosonic states while $|0\rangle^{\dot{\alpha}k}$ and $|0\rangle^{j\dot{A}}$ are fermionic states.

Decomposing $|0\rangle^{jk}$ into its symmetric, anti-symmetric, and trace parts, one finds a graviton g^{jk}, a 'Kalb-Ramond' field B^{jk}, and a scalar dilaton field ϕ. Decomposing $|0\rangle^{j\dot{A}}$ and $|0\rangle^{\dot{\alpha}j}$, one finds two gravitinos, $\psi^{\dot{\alpha}}_j$ and $\psi^{\dot{A}}_j$, and two dilatinos, χ^α and χ^A. Decomposing $|0\rangle^{\dot{\alpha}\dot{\beta}}$ for the Type IIA superstring into $\sigma^j_{\dot{\alpha}\dot{\beta}}|0\rangle^{\dot{\alpha}\dot{\beta}}$ and $\sigma^{jkl}_{\dot{\alpha}\dot{\beta}}|0\rangle^{\dot{\alpha}\dot{\beta}}$, one finds a one-form A^j and an anti-symmetric three-form A^{jkl}. Finally, decomposing $|0\rangle^{\dot{\alpha}\dot{\beta}}$ for the Type IIB superstring into $|0\rangle^{\dot{\alpha}\dot{\alpha}}$, $\sigma^{jk}_{\dot{\alpha}\dot{\beta}}|0\rangle^{\dot{\alpha}\dot{\beta}}$ and $\sigma^{jklm}_{\dot{\alpha}\dot{\beta}}|0\rangle^{\dot{\alpha}\dot{\beta}}$, one finds a scalar A, an anti-symmetric two-form A^{jk}, and a self-dual anti-symmetric four-form A^{jklm} where self-dual means that $\epsilon^{j_1\cdots j_8} A^{j_5\cdots j_8} = 70\, A^{j_1\cdots j_4}$.

These massless states of the superstring are the same as the states of Type IIA and Type IIB supergravity in ten dimensions. However, of course, the superstring also includes an infinite set of massive fields which are not present in pure supergravity theories.

5. Compactification and T-Duality

Lorentz invariance and unitarity imply that superstrings propagate in ten spacetime dimensions, so one needs an explanation for the fact that only four spacetime dimensions are experimentally observable. One possible explanation is that six of the nine spatial directions are constrained to lie on small circles of radius R.

It will be shown here that string theory, unlike point-particle theory, predicts a symmetry called T-duality which relates compactification on a circle of radius R to compactification on a circle of radius $(RT)^{-1}$ where T is the string tension. This means that the radius of compactification can always be chosen larger that $T^{-\frac{1}{2}}$, which has important implications for gravity at the Planck scale since $T^{-\frac{1}{2}}$ is approximately 10^{-32} cm.

First, note that the wave-function $e^{iP^\mu x_\mu}$ should be single-valued when $x^9 \to x^9 + 2\pi R$ if x^9 is a compactified direction. So the momentum P^9 must be equal to nR^{-1} for some integer n.

Next, note that $x^9(\tau, \sigma + 2\pi)$ must equal $x^9(\tau, \sigma) + 2\pi mR$ where m is an integer which counts the number of times that the closed string winds around the compactified direction. This means that the solution to the equation of motion of (2.4) is

$$x^9(\tau, \sigma) = x_0^9 + \frac{n\tau}{TR} + \sigma mR + \sum_{N=-\infty, N\neq 0}^{\infty} (a_N^9 e^{iN(\tau+\sigma)} + \tilde{a}_N^9 e^{iN(\tau-\sigma)}). \qquad (5.1)$$

Plugging into the M^2 formula coming from (2.7), one learns that

$$M_9^2 = (P_0)^2 - (P_1)^2 - ... - (P_8)^2 = P_\mu P^\mu + (P_9)^2 = M_{10}^2 + (\frac{n}{R})^2 \qquad (5.2)$$

$$= (\frac{n}{R})^2 + (mTR)^2 + 2T \sum_{N>0} \sum_{j,\alpha,A=1}^{8} N(n_N^j + \tilde{n}_N^j + m_N^\alpha + \tilde{m}_N^A)$$

where M_9 is the mass measured by a nine-dimensional observer, M_{10} is the mass measured by a ten-dimensional observer, and the $(mTR)^2$ term comes from the $T^2 \partial_\sigma x^\mu \partial_\sigma x_\mu$ contribution to M_{10}^2.

It is easy to see from (5.2) that the nine-dimensional mass spectrum is invariant under switching R with $(TR)^{-1}$ if one also switches momentum excitations n with winding-mode excitations m. Note that T-duality, unlike S-duality discussed in the following secton, does not transform the string coupling constant and can therefore be verified perturbatively.

For the Type II superstring, T-duality states that the Type IIA superstring compactified on a circle of radius R is equivalent to the Type IIB superstring compactified on a circle of radius $(TR)^{-1}$. The reason Type IIA and Type IIB switch places is that switching momentum excitations with winding excitations is only a symmetry of the Type II superstring if $\tilde{\theta}^A$ switches its SO(8) chirality.

6. D=11 Supergravity and M-Theory

Eleven dimensions is the maximum dimension in which gravity can be supersymmetrized in a consistent manner. The bosonic fields of D=11 supergravity are a graviton $\hat{g}_{MN}$ and an anti-symmetric three-form $\hat{A}_{MNP}$ where $M = 0$ to 10. Although this supergravity theory is not renormalizable, its classical action can be constructed and the bosonic contribution to this action is

$$S_{11} = \frac{1}{\lambda^2} \int d^{11}x [\sqrt{\det \hat{g}} (\hat{R} + \hat{F}_{MNPQ}\hat{F}^{MNPQ}) \tag{6.1}$$

$$+\epsilon^{M_1...M_{11}} \hat{F}_{M_1...M_4} \hat{F}_{M_5...M_8} \hat{A}_{M_9 M_{10} M_{11}}]$$

where λ is the gravitatonal coupling constant and $\hat{F}_{MNPQ} = \partial_{[M}\hat{A}_{NPQ]}$ is the field-strength for $\hat{A}_{MNP}$.

After compactification on a circle of radius R_{10}, these fields reduce to the massless bosonic fields of the Type IIA superstring, $[g_{\mu\nu}, B_{\mu\nu}, \phi, A_\mu, A_{\mu\nu\rho}]$ where $\hat{g}_{\mu\nu} = e^{-2\phi/3}g_{\mu\nu}$, $\hat{g}_{10\ 10} = e^{4\phi/3}$, $\hat{g}_{\mu\ 10} = e^{4\phi/3}A_\mu$, $\hat{A}_{\mu\nu\ 10} = B_{\mu\nu}$ and $\hat{A}_{\mu\nu\rho} = A_{\mu\nu\rho}$. With this identification, the Einstein-Hilbert part of the D=11 action $\frac{1}{\lambda^2} \int d^{11}x\sqrt{\det \hat{g}}\hat{R}$ reduces to $\frac{1}{\lambda^2} \int d^{10}xe^{-2\phi}\sqrt{\det g}\hat{R}$. This means that the string coupling constant can be absorbed into a redefinition of $\phi \to \phi + log\lambda$. After this redefinition, the vacuum expectation value for e^ϕ becomes $\langle e^\phi \rangle = \lambda$.

Since the compactification radius is proportional to $\sqrt{\hat{g}_{10\ 10}} = e^{2\phi/3}$, R_{10} is proportional to $\lambda^{2/3}$. Therefore, the Type IIA superstring at weak coupling and low energies (i.e. the massless sector with $\lambda << 1$) is equivalent to compactification of $D = 11$ supergravity on a circle of small radius.

However, the Type IIA superstring is a renormalizable theory, so it also makes sense at high energies. This suggests that there is a renormalizable version of D=11 supergravity with massive fields which makes sense at high-energies. This eleven-dimensional theory is called M-theory and it will now be shown how eleven-dimensional Lorentz invariance of M-theory implies a strong-weak duality of the Type IIB superstring.

The classical low-energy effective action for the Type IIB superstring (i.e. the classical action for ten-dimensional Type IIB supergravity) is known to contain a classical symmetry called S-duality which transforms the massless bosonic Type IIB fields as:

$$\rho \to \frac{a\rho + b}{c\rho + d}, \quad B_{\mu\nu} \to aB_{\mu\nu} + bA_{\mu\nu}, \quad A_{\mu\nu} \to cB_{\mu\nu} + dA_{\mu\nu} \tag{6.2}$$

$$g_{\mu\nu} \to g_{\mu\nu}, \quad A_{\mu\nu\rho\sigma} \to A_{\mu\nu\rho\sigma}$$

where $\rho = A + ie^{-\phi}$ and a, b, c, d are integers satisfying $ad - bc = 1$.

When $a = d = A(x) = 0$ and $b = -c = 1$, this S-duality symmetry transforms $e^{-\phi}$ to e^ϕ, and since $\langle e^\phi \rangle = \lambda$, it takes $\lambda \to \lambda^{-1}$ which switches strong and weak coupling. This strong-weak duality symmetry of the classical action can be proven to be a symmetry of the full quantum Type IIB superstring action using the following argument:

Suppose one compactifies two of the eleven dimensions of M-theory on small circles of radius R_1 and R_2. If R_1 is identified with the eleventh dimension, this corresponds to a Type IIA superstring with $\lambda = (R_1/R_2)^{3/2}$ which is compactified on a small circle of radius R_2. By T-duality, this corresponds to a Type IIB superstring with $\lambda = (R_1/R_2)^{3/2}$ which is compactified on a large circle of radius $(TR_2)^{-1}$.

But by eleven-dimensional Lorentz covariance of M-theory, one could also have identified R_2 with the eleventh dimension. In this case, the M-theory compactification corresponds to a Type IIA superstring with $\lambda' = (R_2/R_1)^{3/2}$ which is compactified on a small circle of radius R_1. By T-duality, this corresponds to a Type IIB superstring with $\lambda' = (R_2/R_1)^{3/2}$ which is compactified on a large circle of radius $(TR_1)^{-1}$.

If $R_1 \to 0$ and $R_2 \to 0$ with $R_1/R_2 = C$ held fixed, the two Type IIB superstrings become uncompactified but their coupling constants remain fixed at the value $\lambda = \lambda'^{-1} = C^{3/2}$. Since these two descriptions come from the same compactification of M-theory, the uncompactified Type IIB superstring is invariant under an S-duality symmetry which exchanges λ and λ^{-1}, and therefore exchanges strong and weak couplings..